Problem Set 2 Solutions – Relativity (G0I36A)

1 η, δ, ∂, and d

1.) ∂^μ is not the same operator as ∂_μ, since raising an index via the metric tensor results in a different tensor. Explicitly, the components of $\partial_\mu\phi$ are

$$\partial_\mu\phi = \frac{\partial\phi}{\partial x^\mu} = \left(\frac{\partial\phi}{\partial x^0}\,,\,\frac{\partial\phi}{\partial x^1}\,,\,\frac{\partial\phi}{\partial x^2}\,,\,\frac{\partial\phi}{\partial x^3}\right)\,, \tag{1.1}$$

while the components of $\partial^\mu\phi = \eta^{\mu\nu}\partial_\nu\phi$ are given by

$$\partial^\mu\phi = \eta^{\mu\nu}\frac{\partial\phi}{\partial x^\nu} = \left(-\frac{\partial\phi}{\partial x^0}\,,\,\frac{\partial\phi}{\partial x^1}\,,\,\frac{\partial\phi}{\partial x^2}\,,\,\frac{\partial\phi}{\partial x^3}\right)\,. \tag{1.2}$$

Due to the minus sign in the $(0,0)$ component of the Minkowski metric tensor $\eta_{\mu\nu}$, there is a minus sign in the zero component of $\partial^\mu\phi$ as compared to the zero component of $\partial_\mu\phi$.

2.) The function $f(x^\mu)$ is given by

$$f(x^\mu) = atx + b(y^2 - xz)\,. \tag{1.3}$$

The components of the derivative $\partial_\mu f$ are easy to write out:

$$\partial_\mu f = \left(\frac{1}{c}\frac{\partial f}{\partial t}\,,\,\frac{\partial f}{\partial x}\,,\,\frac{\partial f}{\partial y}\,,\,\frac{\partial f}{\partial z}\right) = \left(\frac{ax}{c}\,,\,at - bz\,,\,2by\,,\,-bx\right)\,. \tag{1.4}$$

Equivalently, we could also write this as

$$\partial_0 f = \frac{ax}{c}\,,\quad \partial_1 f = at - bz\,,\quad \partial_2 f = 2by\,,\quad \partial_3 f = -bx\,. \tag{1.5}$$

Remember, the coordinate $x^0 = ct$, where c is the (constant) speed of light, so $\frac{\partial}{\partial x^0} = \frac{1}{c}\frac{\partial}{\partial t}$, which is responsible for the factor of c in the denominator of $\partial_0 f$.

Now, we compute $\partial^\mu\partial_\mu f$ as follows:

$$\begin{aligned}
\partial^\mu\partial_\mu f &= \eta^{\mu\nu}\partial_\nu\partial_\mu f \\
&= \eta^{00}\partial_0\partial_0 f + \eta^{11}\partial_1\partial_1 f + \eta^{22}\partial_2\partial_2 f + \eta^{33}\partial_3\partial_3 f \\
&= -\partial_0^2 f + \partial_1^2 f + \partial_2^2 f + \partial_3^2 f \\
&= -\frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} + \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \\
&= 2b\,.
\end{aligned} \tag{1.6}$$

In the end, the only term that was non-zero was $\partial_2\partial_2 f = 2b$. It is also useful to note that since the Minkowski metric $\eta_{\mu\nu}$ is diagonal, the Laplacian $\partial^\mu\partial_\mu$ ended up having no mixed derivatives. Additionally, since we can raise/lower pairs of contracted indices freely, $\partial^\mu\partial_\mu f$ is exactly the same thing as $\partial_\mu\partial^\mu f$.

We now move on to computing $(\partial_\mu f)(\partial^\mu f)$ as follows:

$$\begin{aligned}
(\partial_\mu f)(\partial^\mu f) &= \eta^{\mu\nu}(\partial_\mu f)(\partial_\nu f) \\
&= \eta^{00}(\partial_0 f)(\partial_0 f) + \eta^{11}(\partial_1 f)(\partial_1 f) + \eta^{22}(\partial_2 f)(\partial_2 f) + \eta^{33}(\partial_3 f)(\partial_3 f) \\
&= -(\partial_0 f)^2 + (\partial_1 f)^2 + (\partial_2 f)^2 + (\partial_3 f)^2 \\
&= -\left(\frac{ax}{c}\right)^2 + (at - bz)^2 + (2by)^2 + (-bx)^2 \\
&= -\frac{a^2x^2}{c^2} + (at - bz)^2 + 4b^2y^2 + b^2x^2 \ .
\end{aligned} \tag{1.7}$$

Clearly, by comparing (1.6) and (1.7), we can see that $\partial^\mu\partial_\mu f \neq (\partial_\mu f)(\partial^\mu f)$. This makes sense; the former is second-order derivative operator acting on a single instance of the function f, while the latter is two instances of first-order derivatives of f being contracted with one another. This is the special relativity generalization of the fact you learned in calculus that $\vec{\nabla} \cdot \vec{\nabla} f \neq \vec{\nabla} f \cdot \vec{\nabla} f$.

3.) First, we explicitly write out the sum over the contracted index ρ as

$$\eta^{\mu\rho}\eta_{\nu\rho} = \eta^{\mu 0}\eta_{\nu 0} + \eta^{\mu 1}\eta_{\nu 1} + \ldots + \eta^{\mu(d-1)}\eta_{\nu(d-1)} \ , \tag{1.8}$$

since we are considering a d-dimensional spacetime for which the index μ ranges over all integers from 0 to $d-1$. Since the Minkowski metric $\eta_{\mu\nu}$ and its inverse $\eta^{\mu\nu}$ are diagonal, the only way to make any term in this sum non-zero is to have $\mu = \nu$. That is, if $\mu \neq \nu$, then each term in this sum is zero, and so we have

$$\eta^{\mu\rho}\eta_{\nu\rho} = 0 \quad \text{if } \mu \neq \nu \ . \tag{1.9}$$

Now, if $\mu = \nu = 0$, then (again using the diagonal nature of the metric tensor) only the first term is non-zero; the result will be $\eta^{00}\eta_{00} = (-1)\cdot(-1) = 1$. Similarly, if $\mu = \nu = 1$, then only the second term in the above sum is non-zero, and it is given by $\eta^{11}\eta_{11} = 1 \cdot 1 = 1$. If we repeat this logic for all possible values of $\mu = \nu$, we find that only one term in the sum will contribute, and its contribution will be exactly 1. Thus, we have

$$\eta^{\mu\rho}\eta_{\nu\rho} = 1 \quad \text{if } \mu = \nu \ . \tag{1.10}$$

Putting together (1.9) and (1.10), we have thus shown that

$$\eta^{\mu\rho}\eta_{\nu\rho} = \begin{cases} 1 & \mu = \nu \\ 0 & \mu \neq \nu \end{cases} \ . \tag{1.11}$$

The right-hand side of this expression is precisely the definition of the Kronecker delta symbol δ^μ_ν, and so we have proven that $\eta^{\mu\rho}\eta_{\nu\rho} = \delta^\mu_\nu$.

Now, if we use this result and fully contract both indices, we find that

$$\eta^{\mu\nu}\eta_{\mu\nu} = \delta^\mu_\mu \ . \tag{1.12}$$

The μ index is a dummy index that we must sum over, and we can explicitly write out this sum as

$$\eta^{\mu\nu}\eta_{\mu\nu} = \delta^0_0 + \delta^1_1 + \ldots + \delta^{(d-1)}_{(d-1)} \ . \tag{1.13}$$

Each term in this sum is equal to 1, since the top and bottom indices on the Kronecker delta symbol match, and there are d terms total (one for each spacetime dimension). The total sum is therefore simply $1 \cdot d = d$, and thus $\eta^{\mu\nu}\eta_{\mu\nu} = d$.

4.) Let's consider the right-hand side of the given expression, $\delta^{\mu_1}_{\nu} T^{\nu\mu_2\cdots\mu_n}$. In this tensor, ν is a dummy index that is summed over. Since the Kronecker delta symbol $\delta^{\mu_1}_{\nu}$ is defined to be non-zero if and only if $\mu_1 = \nu$, the sum over the dummy index ν will be zero on all terms *except* the term in the sum where $\nu = \mu_1$. Moreover, the Kronecker delta symbol is defined to be 1 when $\nu = \mu_1$, and so for the term in the sum where $\mu_1 = \nu$ we can simply replace ν with μ_1 and get rid of the δ symbol. Thus, we find that $\delta^{\mu_1}_{\nu} T^{\nu\mu_2\cdots\mu_n} = T^{\mu_1\cdots\mu_n}$, as desired.

2 Manipulating the Einstein Equation

5.) You should first note that the Ricci scalar R does not explicitly appear in the equation we want to prove. This tells you that we want to solve for R and then plug it back into the Einstein equation. To do this, we need to take the full tensor equation and turn it into a scalar equation, which we can in turn use to solve for R. So, will take the trace of the Einstein equation, i.e. contract it with $g^{\mu\nu}$ on both the left- and right-hand sides:

$$g^{\mu\nu} R_{\mu\nu} - \frac{1}{2} g^{\mu\nu} g_{\mu\nu} R + g^{\mu\nu} g_{\mu\nu} \Lambda = 8\pi G_N g^{\mu\nu} T_{\mu\nu} \ . \tag{2.1}$$

The first term is simply the Ricci scalar $R = g_{\mu\nu} R^{\mu\nu} = g^{\mu\nu} R_{\mu\nu}$, the trace of the Ricci tensor. For the second and third terms, we need to first recall that $g^{\mu\nu} g_{\rho\nu} = \delta^{\mu}_{\rho}$, by definition of $g^{\mu\nu}$ being the inverse metric. Then, since we are in a d-dimensional spacetime, the Kronecker delta symbol has a trace of d:

$$g^{\mu\nu} g_{\mu\nu} = \delta^{\mu}_{\mu} = d \ . \tag{2.2}$$

Finally, since the inverse metric can be used to raise indices, we can write $g^{\mu\nu} T_{\mu\nu} = T_{\mu}{}^{\mu}$. Thus, (2.1) becomes

$$\left(1 - \frac{d}{2}\right) R + d\Lambda = 8\pi G_N T_{\mu}{}^{\mu} \ . \tag{2.3}$$

Assuming $d > 2$, the coefficient of R in this equation is non-zero, and so we can solve for R:

$$R = \frac{2}{2-d} \left(8\pi G_N T_{\mu}{}^{\mu} - d\Lambda\right) \ . \tag{2.4}$$

With this in mind, we can rewrite the Einstein equation as follows:

$$\begin{aligned} R_{\mu\nu} &= 8\pi G T_{\mu\nu} + \frac{1}{2} g_{\mu\nu} R - g_{\mu\nu} \Lambda \\ &= 8\pi G_N T_{\mu\nu} + \frac{1}{2-d} g_{\mu\nu} \left(8\pi G_N T_{\rho}{}^{\rho} - d\Lambda\right) - g_{\mu\nu} \Lambda \\ &= 8\pi G_N T_{\mu\nu} + \frac{1}{2-d} g_{\mu\nu} \left(8\pi G_N T_{\rho}{}^{\rho} - d\Lambda - (2-d)\Lambda\right) \\ &= 8\pi G_N T_{\mu\nu} + \frac{2}{2-d} g_{\mu\nu} \left(4\pi G_N T_{\rho}{}^{\rho} - \Lambda\right) \ . \end{aligned} \tag{2.5}$$

This is the final expression we were looking for.

6.) When $d = 2$, the coefficient of the Ricci scalar in (2.3) vanishes, and so we can no longer solve for R. Instead, this relation now tells us that

$$\Lambda = 4\pi G_N T_{\mu}{}^{\mu} \ . \tag{2.6}$$

If we plug this back in for Λ in the original Einstein equation, then we find that

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G_N \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T_{\rho}{}^{\rho}\right) \ . \tag{2.7}$$

Interestingly, both the left- and right-hand sides of this expression become very similar; they can both be written as a symmetric tensor, with part of its trace subtracted. This is no accident; the Einstein equation actually develops some interesting problems in two dimensions. If you're curious, come ask me about it sometime!

7.) Since $R_{\mu\nu} = kg_{\mu\nu}$, we also know that $R = g^{\mu\nu}R_{\mu\nu} = kg^{\mu\nu}g_{\mu\nu} = dk$. The Einstein equation becomes

$$\frac{2-d}{2}kg_{\mu\nu} + g_{\mu\nu}\Lambda = 8\pi G_N T_{\mu\nu} \ . \tag{2.8}$$

When the stress-energy tensor $T_{\mu\nu}$ vanishes, this equation reduces to

$$g_{\mu\nu}\left(\frac{2-d}{2}k + \Lambda\right) = 0 \ . \tag{2.9}$$

It doesn't make sense to solve this by setting $g_{\mu\nu} = 0$; this would mean that we don't have a space-time on which we can understand physics. Instead, the term inside the parentheses must vanish. When $d = 2$, the coefficient of k vanishes, and we are left with $\Lambda = 0$. When $d \neq 2$, though, we can solve for k to find that

$$k = \frac{2\Lambda}{d-2} \ . \tag{2.10}$$

So, Einstein metrics in $d > 2$ dimensions necessarily have this constant k proportional to the cosmological constant Λ, when there is no stress-energy tensor.

3 Lorentz Transformations

8.) Let's consider the tensor on the right-hand side of the Lorentz transformation condition, ${\Lambda^\mu}_\rho {\Lambda^\nu}_\sigma \eta_{\mu\nu}$. The free indices are ρ and σ, so we will have to go over all values possible for ρ and σ and compare to $\eta_{\rho\sigma}$ to see if the Lorentz transformation condition is satisfied.

We will first compute the $\rho = 0$, $\sigma = 0$ component. Since ${\Lambda^0}_0$ and ${\Lambda^2}_0$ are both non-zero while ${\Lambda^1}_0$ and ${\Lambda^3}_0$ are both zero, the sum over the dummy indices μ and ν will yield four potentially non-zero terms:

$${\Lambda^\mu}_0{\Lambda^\nu}_0\eta_{\mu\nu} = {\Lambda^0}_0{\Lambda^0}_0\eta_{00} + {\Lambda^2}_0{\Lambda^0}_0\eta_{20} + {\Lambda^0}_0{\Lambda^2}_0\eta_{02} + {\Lambda^2}_0{\Lambda^2}_0\eta_{22} \ . \tag{3.1}$$

The metric tensor is diagonal, and so $\eta_{20} = \eta_{02} = 0$. And, we also know by its definition that $\eta_{00} = -1$, $\eta_{22} = 1$. Thus we find

$${\Lambda^\mu}_0{\Lambda^\nu}_0\eta_{\mu\nu} = -({\Lambda^0}_0)^2 + ({\Lambda^2}_0)^2 = -(1+a^2) + a^2 = -1 \ . \tag{3.2}$$

Then, since we know that $\eta_{00} = -1$, we thus find that the Lorentz transformation condition is satisfied for the $\rho = 0$, $\sigma = 0$ component.

Another term we will compute explicitly is the $\rho = 0$, $\sigma = 2$ component:

$$\begin{aligned}
{\Lambda^\mu}_0{\Lambda^\nu}_2\eta_{\mu\nu} &= {\Lambda^0}_0{\Lambda^0}_2\eta_{00} + {\Lambda^2}_0{\Lambda^0}_2\eta_{20} + {\Lambda^0}_0{\Lambda^2}_2\eta_{02} + {\Lambda^2}_0{\Lambda^2}_2\eta_{22} \\
&= {\Lambda^0}_0{\Lambda^0}_2\eta_{00} + {\Lambda^2}_0{\Lambda^2}_2\eta_{22} \\
&= -{\Lambda^0}_0{\Lambda^0}_2 + {\Lambda^2}_0{\Lambda^2}_2 \\
&= -a\sqrt{1+a^2} + a\sqrt{1+a^2} \\
&= 0 \ .
\end{aligned} \tag{3.3}$$

Thus this vanishes, just as $\eta_{02} = 0$. So, again we have a match.

We could go on computing all 16 components explicitly, but it's pretty straightforward and simple to do, so I'll leave it as an exercise for you guys to do. Note that you don't actually need to do 16 calculations; symmetry cuts down the number of independent terms to 10, and most of them are obviously zero. At the end of the day, you will find that $\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma \eta_{\mu\nu} = \eta_{\rho\sigma}$ for all values of ρ and σ.

9.) Note: this problem was worded a little unclearly, and I apologize for any confusion. By $\Lambda_\mu{}^\nu$, what I meant was simply $\eta_{\mu\rho}\eta^{\nu\sigma}\Lambda^\rho{}_\sigma$. However, it is true that Lorentz transformations are not actually tensors, and so $\Lambda_\mu{}^\nu$ is not actually well-defined, and some books actually denote $\Lambda_\mu{}^\nu$ and $\Lambda^\nu{}_\mu$ to be the same thing. So, let's just (for the sake of this problem) define $\Lambda_\mu{}^\nu \equiv \eta_{\mu\rho}\eta^{\nu\sigma}\Lambda^\rho{}_\sigma$, ignore the fact that the left-hand side is not actually well-defined, and go ahead anyways.

First, as an example, let's compute the component $\Lambda_0{}^0$:

$$\Lambda_0{}^0 = \eta_{0\rho}\eta^{0\sigma}\Lambda^\rho{}_\sigma = \eta_{00}\eta^{00}\Lambda^0{}_0 = (-1)\cdot(-1)\Lambda^0{}_0 = \Lambda^0{}_0 \ . \tag{3.4}$$

The second equality came from summing over the dummy indices ρ and σ while also noting that the metric tensor and its inverse are diagonal, and so the only non-zero term in the sum is the one where $\rho = \sigma = 0$. From there, it's just simple plug-and-chug. The two minus signs coming from the metric tensor cancel out, and so we are left with $\Lambda_0{}^0 = \Lambda^0{}_0$. In fact, you should be able to see that the same thing will happen for all of the components on the diagonal, i.e. $\Lambda_1{}^1 = \Lambda^1{}_1$, $\Lambda_2{}^2 = \Lambda^2{}_2$, and $\Lambda_3{}^3 = \Lambda^3{}_3$, the only difference being that you will get $1 \cdot 1$ from the metric components instead of $(-1)\cdot(-1)$.

However, we will not find that all components are exactly the same. Consider, for example, the $\mu = 0$, $\nu = 2$ component:

$$\Lambda_0{}^2 = \eta_{0\rho}\eta^{2\sigma}\Lambda^\rho{}_\sigma = \eta_{00}\eta^{22}\Lambda^0{}_2 = (-1)\cdot(1)\Lambda^0{}_2 = -\Lambda^0{}_2 \ . \tag{3.5}$$

That is, the metric factors are such that we pick up an overall minus sign when there is exactly one time-like and one space-like component. Similarly, you should also see easily that $\Lambda_0{}^1 = -\Lambda^0_1$ and $\Lambda_0{}^3 = -\Lambda^0{}_3$. Additionally, you will also pick up a minus sign when $\nu = 0$ and $\mu = 2$:

$$\Lambda_2{}^0 = \eta_{2\rho}\eta^{0\sigma}\Lambda^\rho{}_\sigma = \eta_{22}\eta^{00}\Lambda^2{}_0 = (1)\cdot(-1)\Lambda^2{}_0 = -\Lambda^2{}_0 \ . \tag{3.6}$$

And, similarly, you will find this same minus sign when $\mu = 1$ or $\mu = 3$ as well.

When you put everything together, you should find that you get a relative minus sign if either $\mu = 0$ or $\nu = 0$, but not both. The matrix form of $\Lambda_\mu{}^\nu$ will thus end up looking like:

$$\Lambda_\mu{}^\nu = \begin{pmatrix} \sqrt{1+a^2} & 0 & -a & 0 \\ 0 & 1 & 0 & 0 \\ -a & 0 & \sqrt{1+a^2} & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \ . \tag{3.7}$$

Due to the minus signs on the $\mu = 0$, $\nu = 2$ components and $\mu = 2$, $\nu = 0$ components, this is not the same matrix form as the one given for $\Lambda^\mu{}_\nu$. Note that you could have also gotten this matrix form by computing the matrix product $\eta^{-1}\cdot\Lambda\cdot\eta$, though I think going through the actual index gymnastics is more enlightening and better practice.

10.) **Bonus:** Under a Lorentz transformation, the coordinates transform as follows:

$$x^\mu \to x^{\mu'} = \Lambda^{\mu'}{}_\nu x^\nu \ . \tag{3.8}$$

For the given Lorentz transformation, this tells us that the transformed coordinates are such that

$$\begin{pmatrix} x^{0'} \\ x^{1'} \\ x^{2'} \\ x^{3'} \end{pmatrix} = \begin{pmatrix} \sqrt{1+a^2} & 0 & a & 0 \\ 0 & 1 & 0 & 0 \\ a & 0 & \sqrt{1+a^2} & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} \sqrt{1+a^2}\, x^0 + ax^2 \\ x^1 \\ ax^0 + \sqrt{1+a^2}\, x^2 \\ -x^3 \end{pmatrix} \ . \tag{3.9}$$

Thus, we can see that our Lorentz transformation mixes up the x^0 and x^2 coordinates, leaves x^1 alone, and flips the sign of x^3. In fact, the way that the x^0 and x^2 coordinates are mixed is exactly the usual boost in the x^2 direction; you can see this by first redefining

$$a = -\sinh\phi \ . \tag{3.10}$$

The transformed coordinates can then be written as

$$\begin{pmatrix} x^{0'} \\ x^{1'} \\ x^{2'} \\ x^{3'} \end{pmatrix} = \begin{pmatrix} \cosh\phi\, x^0 - \sinh\phi\, x^2 \\ x^1 \\ -\sinh\phi\, x^0 + \cosh\phi\, x^2 \\ -x^3 \end{pmatrix} \ . \tag{3.11}$$

If we then let $\phi = \tanh^{-1} v$, then this finally can be written as

$$\begin{pmatrix} x^{0'} \\ x^{1'} \\ x^{2'} \\ x^{3'} \end{pmatrix} = \begin{pmatrix} \gamma\left(x^0 - vx^2\right) \\ x^1 \\ \gamma\left(x^2 - vx^0\right) \\ -x^3 \end{pmatrix} \ , \quad \gamma \equiv \frac{1}{\sqrt{1-v^2}} \ . \tag{3.12}$$

We can then study the $x^{0'}$ and $x^{2'}$ components and notice that they are given by exactly the familiar boost equation from special relativity. Therefore I have given you a Lorentz transformation that boosts our inertial frame in the x^2 direction while also flipping the x^3 direction (i.e. doing a parity transformation along the x^3 direction).

The full group of Lorentz transformations is $O(3,1)$. The $SO(3,1) \subset O(3,1)$ subset of this is referred to as the proper Lorentz group; it comprises all Lorentz transformations that satisfy the property $\det(\Lambda) = 1$. The further subgroup $SO(3,1)^\uparrow \subset SO(3,1)$ is the restricted Lorentz group, and it comprises all Lorentz transformations that have $\Lambda^0{}_0 \geq 1$ as well as the $\det(\Lambda) = 1$ condition.

If you compute the determinant of the given Lorentz transformation, you will find $\det(\Lambda) = -1$, and thus it does not fall into either of the subgroups above. The offender responsible for this is the -1 entry on the $\Lambda^3{}_3$ component. Thus we can see that parity transformations are not allowed in the proper Lorentz group.

11.) **Bonus:** No, the sum of Lorentz transformations is in general not guaranteed to also be a Lorentz transformation. There are many examples you can cook up. One particular one I like is letting Λ_1 be the identity Lorentz transformation and Λ_2 be the Lorentz transformation associated with time-reversal and parity transformations. In component form, these look like

$$(\Lambda_1)^\mu{}_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \ , \quad (\Lambda_2)^\mu{}_\nu = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \ . \tag{3.13}$$

(If you're not sure why these are Lorentz transformations, I suggest working out the Lorentz transformation condition explicitly!) The sum $\Lambda^\mu{}_\nu$ of these is simply the zero matrix, which obviously does not satisfy the Lorentz transformation condition; such a matrix causes $\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma \eta_{\mu\nu}$ to be zero for all components, and thus it cannot match $\eta_{\rho\sigma}$.

4 Actions and the Euler-Lagrange Equations

12.) The only coordinate is t, the field is $h(t)$, and the Lagrangian is

$$\mathcal{L} = \frac{1}{2} m (\partial_t h)^2 - mgh \ , \tag{4.1}$$

where ∂_t is shorthand for $\frac{\partial}{\partial t}$. The Euler-Lagrange equation for this system therefore takes the form

$$\frac{\partial \mathcal{L}}{\partial h} - \partial_t \left(\frac{\partial \mathcal{L}}{\partial(\partial_t h)} \right) = 0 \ . \tag{4.2}$$

For the purposes of taking derivatives, note that h and $\partial_t h$ are treated as independent quantities. This means that

$$\frac{\partial \mathcal{L}}{\partial h} = \frac{\partial}{\partial h} (-mgh) = -mg \ . \tag{4.3}$$

Next, we compute that the derivative with respect to $\partial_t h$ is

$$\frac{\partial \mathcal{L}}{\partial(\partial_t h)} = \frac{\partial}{\partial(\partial_t h)} \left(\frac{1}{2} m (\partial_t h)^2 \right) = m \partial_t h \ . \tag{4.4}$$

Note that all of our usual rules for derivatives apply when deriving the Euler-Lagrange equation. In particular, we used here the fact that $\partial_x(x^2) = 2x$.

Plugging (4.3) and (4.4) into (4.2), the Euler-Lagrange equation becomes

$$0 = (-mg) - \partial_t (m \partial_t h) = -mg - m\partial_t^2 h \ , \tag{4.5}$$

where in the second equality we pulled the constant mass m outside the derivative. If we go back to classical mechanics and recall that the vertical acceleration of our object is defined by $a = \frac{\partial^2 h}{\partial t^2}$, then our equation of motion thus simply tells us that

$$\boxed{a = -g} \ . \tag{4.6}$$

Our Euler-Lagrange equation thus tells us that $a = -g$ for massive objects in a uniform gravitational field pointing downwards, which is of course familiar from any introductory mechanics course.

13.) Remember, we have to treat ϕ and $\partial_\mu \phi$ as independent quantities. This means we don't have to worry about the first term in the Lagrangian, i.e. the term with derivatives acting on ϕ, when differentiating $\mathcal{L}$ with respect to ϕ. This derivative is easy to compute:

$$\frac{\partial \mathcal{L}}{\partial \phi} = -m^2 \phi \ . \tag{4.7}$$

14.) This problem asks you to motivate the following identity for differentiation with respect to vector fields:

$$\frac{\partial A_\nu}{\partial A_\mu} = \delta^\mu_\nu \ . \tag{4.8}$$

Where does this relation come from? Well, we have to consider $A_\mu = (A_0, A_1, A_2, A_3)$ as a vector of four independent objects. Since A_0 and A_1, for instance, are independent, this means that

$$\frac{\partial A_0}{\partial A_1} = 0 \ , \quad \frac{\partial A_1}{\partial A_0} = 0 \ , \quad \frac{\partial A_1}{\partial A_1} = 1 \ , \quad \text{etc.} \tag{4.9}$$

So, we get zero unless the two indices match, in which case we get one. This is precisely the relation (4.8). This holds for any vector field, including $\partial_\mu \phi$.

15.) Making sure to rewrite the dummy indices in $\mathcal{L}$ appropriately so that there are no μ's (so we can avoid any index confusions), we find that

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = -\frac{1}{2}\eta^{\nu\rho}\frac{\partial(\partial_\nu\phi\partial_\rho\phi)}{\partial(\partial_\mu\phi)} = -\frac{1}{2}\eta^{\nu\rho}\left(\delta^\mu_\nu\partial_\rho\phi + \delta^\mu_\rho\partial_\nu\phi\right) \ . \tag{4.10}$$

Note also that we pulled $\eta^{\nu\rho}$ out of the derivative, since it is a constant. If we now contract indices appropriately (i.e. $\eta^{\nu\rho}\delta^\mu_\nu = \eta^{\mu\rho}$), we are left with

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = -\partial^\mu\phi \ , \tag{4.11}$$

where $\partial^\mu = \eta^{\mu\nu}\partial_\nu$.

16.) Plugging (4.7) and (4.11) into the Euler-Lagrange equation, our equation of motion is

$$\boxed{\partial_\mu\partial^\mu\phi - m^2\phi = 0} \ . \tag{4.12}$$

This equation is called the *Klein-Gordon equation*. To understand it, let's write out the operator $\partial_\mu\partial^\mu$ explicitly, remembering that $x^0 = ct$:

$$\partial_\mu\partial^\mu = \eta^{\mu\nu}\partial_\mu\partial_\nu = -\partial_0^2 + \partial_1^2 + \partial_2^2 + \partial_3^2 = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \ . \tag{4.13}$$

Thus, the Klein-Gordon equation can be written explicitly out as

$$\left(-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - m^2\right)\phi = 0 \ . \tag{4.14}$$

You should recognize this as a *wave equation* in three spatial dimensions. In the limit where $m \to 0$, it becomes the ordinary version in which the wave propagates at speed c, but the propagation speed will be in general lower due to the scalar field's mass.

17.) **Bonus:** If $\phi = \phi(t)$, and not of any of the other spacetime coordinates, then $\partial_1\phi = \partial_2\phi = \partial_3\phi = 0$, and so the Klein-Gordon equation above simplifies to

$$\frac{\partial^2\phi}{\partial t^2} = -m^2c^2\phi \ . \tag{4.15}$$

This is a second-order linear differential equation with two independent solutions $\phi_\pm = e^{\pm imct}$. The most general solution to the differential equation will be a linear combination of these two roots:

$$\boxed{\phi = Ae^{imct} + Be^{-imct}} \ , \tag{4.16}$$

for some constants A and B. These constants are in principle determined by specifying boundary conditions for the field ϕ, i.e. specifying what ϕ is at two different times.